Compact operators that commute with a contraction

Karim Kellay, Mohamed Zarrabi

COMPACT OPERATORS THAT COMMUTE WITH A CONTRACTION

K. KELLAY AND M. ZARRABI

ABSTRACT. Let T be a C_0–contraction on a separable Hilbert space. We assume that $I_H - T^*T$ is compact. For a function f holomorphic in the unit disk $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, we show that $f(T)$ is compact if and only if f vanishes on $\sigma(T) \cap \mathbb{T}$, where $\sigma(T)$ is the spectrum of T and $\mathbb{T}$ the unit circle. If f is just a bounded holomorphic function on $\mathbb{D}$ we prove that $f(T)$ is compact if and only if $\lim_{n\to\infty} T^n f(T) = 0$.

1. INTRODUCTION

Let H be a separable Hilbert space, and $\mathcal{L}(H)$ the space of all bounded operators on H. For $T \in \mathcal{L}(H)$, we denote by $\sigma(T)$ the spectrum of T. The Hardy space H^∞ is the set of all bounded and holomorphic functions on $\mathbb{D}$. The spectrum of an inner function θ is defined by

$$\sigma(\theta) = \operatorname{clos} \theta^{-1}(0) \cup \operatorname{supp} \mu,$$

where μ is the singular measure associated to the singular part of θ and $\operatorname{supp} \mu$ is the closed support of μ (see [13], p. 63). A contraction T on H is called a C_0–contraction (or in class C_0) if it is completely nonunitary and there exists a nonzero function $\theta \in \mathrm{H}^\infty$ such that $\theta(T) = 0$, thus there exists a minimal inner function m_T that annihilates T, i.e $m_T(T) = 0$, and we have $\sigma(T) = \sigma(m_T)$ (see [11] p. 117 and [13], p. 71-72). A contraction T is said essentially unitary if $I_H - T^*T$ is compact, where I_H is the identity map on H.

Let T be a C_0–contraction on H, and let $\mathrm{H}^\infty(T) = \{f(T) : f \in \mathrm{H}^\infty\}$. $\mathrm{H}^\infty(T)$ is clearly a subspace of the commutant $\{T\}' = \{A \in \mathcal{L}(H) : AT = TA\}$. In this note we study the question of when $\mathrm{H}^\infty(T)$ contains a nonzero compact operator. B. Sz–Nagy [12], proved that $\{T\}'$ contains always a nonzero compact operator, but there exists a C_0–contraction T such that zero is the unique compact operator contained in $\mathrm{H}^\infty(T)$ (see section 4). Nordgreen [15] proved that if T is an essentially unitary C_0–contraction

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then $\mathrm{H}^\infty(T)$ contains a nonzero compact operator. There are also results about the existence of smooth operators (finite rank, Schatten–von Neuman operators) in $\mathrm{H}^\infty(T)$ (see [18]). It is also shown in the Atzmon's paper [2], that if T is a cyclic completely nonunitary contraction such that $\sigma(T) = \{1\}$ and

$$\log \|T^{-n}\| = O(\sqrt{n}), \; n \to \infty, \tag{1}$$

then $T - I_H$ is compact. Our hope in this paper is to establish results of this kind for more general contractions. Section 2 is devoted to the study of the compactness of $f(T)$ when f is in the disk algebra. We recall that the disk algebra $\mathcal{A}(\mathbb{D})$ is the space of all functions that are holomorphic in $\mathbb{D}$ and continuous on $\mathbb{T}$. We show (Theorem 2.1), that, if $f \in \mathcal{A}(\mathbb{D})$ and if T is a C_0–contraction which is essentially unitary, then $f(T)$ is compact if and only if f vanishes on $\sigma(T) \cap \mathbb{T}$. The main tool used in the proof of this result is the Beurling-Rudin theorem about the characterization of the closed ideals of $\mathcal{A}(\mathbb{D})$. For a large class of C_0–contractions we show (Proposition 2.5) that the condition " T is essentially unitary" is necessary in the above result. As corollary, we obtain that if T is a contraction that is annihilated by a nonzero function in $\mathcal{A}(\mathbb{D})$ and if T is cyclic (or, more generally, of finite multiplicity) then $f(T)$ is compact whenever $f \in \mathcal{A}(\mathbb{D})$ and f vanishes on $\sigma(T) \cap \mathbb{T}$. We notice that an invertible contraction with spectrum reduced to a single point and satisfying condition (1) is necessarily annihilated by a nonzero function in $\mathcal{A}(\mathbb{D})$ (see [1]).

In section 3, we are interested in the compactness of $f(T)$ when $f \in \mathrm{H}^\infty$. Let H^2 be the usual Hardy space on $\mathbb{D}$, θ being an inner function and $\mathrm{K}_\theta = \mathrm{H}^2 \ominus \theta\mathrm{H}^2$. We consider the model operator $T_\theta : \mathrm{K}_\theta \to \mathrm{K}_\theta$ given by $T_\theta g = P_\theta(zg)$, where P_θ stands the orthogonal projection on K_θ. Notice that $m_{T_\theta} = \theta$ and the commutant $\{T_\theta\}'$ of T_θ coincide with $\mathrm{H}^\infty(T_\theta)$. Hartman [7] and Sarason [17] gave a complete characterization of compact operators commuting with T_θ (see [13] p. 182). They showed that

$$T_\theta^n f(T_\theta) \to 0 \iff f(T_\theta) \text{ is compact } \iff f\overline{\theta} \in \mathrm{H}^\infty + C(\mathbb{T}).$$

For C_{00} contractions, that is contractions T such that $T^n x \longrightarrow 0$ and $T^{*n} x \longrightarrow 0$ for every $x \in H$, Muhly gave in [10] (see also [16]) a characterization of functions $f \in \mathrm{H}^\infty$ such that $f(T)$ is compact, in term of the characteristic function of T. With the help of the corona theorem, we show (Theorem 3.4) that if T is an essentially unitary C_0-contraction, then $f(T)$ ($f \in \mathrm{H}^\infty$) is compact if and only if $T^n f(T) \to 0$. We obtain in particular that if $\lim_{r \to 1-} f(rz) = 0$ for every $z \in \sigma(T) \cap \mathbb{T}$, then $f(T)$ is compact.

2. Compactness of $f(T)$ with f in the disk algebra

Let T be a C_0–contraction on H. We will introduce some definitions and results we will need later. We call $\lambda \in \sigma(T)$ a normal eigenvalue if it is an isolated point of $\sigma(T)$ and if the corresponding Riesz projection has a finite rank. We set $\sigma_{np}(T)$ the set of all normal eigenvalues of T. The weakly continue spectrum of T is defined by $\sigma_{wc}(T) = \sigma(T) \setminus \sigma_{np}(T)$ (see [14], p. 113).

Let us suppose furthermore that T is essentially unitary. Since $\mathbb{D} \setminus \sigma(T) \neq \emptyset$, there exists a unitary operator U and a compact operator K such that $T = U + K$ (see [14] p. 115) and we have $\sigma_{wc}(T) = \sigma_{wc}(U) \subset \mathbb{T}$. Since T is in class C_0 the set of eigenvalues of T is $\sigma(T) \cap \mathbb{D}$ (see [13] p. 72). In particular we have $\sigma_{np}(T) \subset \sigma(T) \cap \mathbb{D}$. It follows that $\sigma_{wc}(T) = \sigma(T) \cap \mathbb{T}$ and $\sigma_{np}(T) = \sigma(T) \cap \mathbb{D}$. We deduce also from the above observations that if T is in class C_0 then T is essentially unitary if and only if T^* is too.

Let $\mathcal{I}$ be a closed ideal of $\mathcal{A}(\mathbb{D})$. We denote by $S_{\mathcal{I}}$ the inner factor of $\mathcal{I}$, that is the greatest inner common divisor of all nonzero functions in $\mathcal{I}$ (see [8] p. 85), and we set $Z(\mathcal{I}) = \bigcap_{f \in \mathcal{I}} \{\zeta \in \mathbb{T} : f(\zeta) = 0\}$. For $E \subset \mathbb{T}$ we set $\mathcal{J}(E) = \{f \in \mathcal{A}(\mathbb{D}) : f_{|E} = 0\}$. We shall need the Beurling-Rudin theorem (see [8] p. 85) about the structure of closed ideals of $\mathcal{A}(\mathbb{D})$, which states that every closed ideal $\mathcal{I} \subset \mathcal{A}(\mathbb{D})$ has the form

$$\mathcal{I} = S_{\mathcal{I}} \mathrm{H}^{\infty} \cap \mathcal{J}\big(Z(\mathcal{I})\big).$$

Theorem 2.1. *Let T be an essentially unitary C_0–contraction and let $f \in \mathcal{A}(\mathbb{D})$. The following assertions are equivalents.*

(1) $f(T)$ is compact.
(2) $f = 0$ on $\sigma(T) \cap \mathbb{T}$.

For the proof of this theorem we need the following lemma.

Lemma 2.2. *Let T_1, T_2 be two contractions on H such that $T_1 - T_2$ is compact and $f \in A(\mathbb{D})$. Then $f(T_1)$ is compact if and only if $f(T_2)$ is too.*

Proof. There exists a sequence $(P_n)_n$ of polynomials such that $\|f - P_n\|_\infty \to 0$, where $\|.\|_\infty$ is the supremum norm on $\mathbb{T}$. We set $R_n = f - P_n$. Note that for every n, $P_n(T_2) - P_n(T_1)$ is compact. On the other hand, for $i = 1, 2$, by the von Neumann inequality, we have $\|R_n(T_i)\| \leq \|R_n\|_\infty$. So $\|R_n(T_i)\| \longrightarrow 0$. It follows that

$$\begin{aligned} f(T_2) - f(T_1) &= \lim_{n \to +\infty} \big(P_n(T_2) - P_n(T_1) + R_n(T_2) - R_n(T_1)\big) \\ &= \lim_{n \to +\infty} \big(P_n(T_2) - P_n(T_n)\big). \end{aligned}$$

Thus $f(T_2) - f(T_1)$ is compact, which finishes the proof. $\square$

Proof of Theorem 2.1. (1) $\Rightarrow$ (2) : Let $\mathcal{B}_T$ denote the maximal commutative Banach algebra that contains I_H and T. We have $\sigma(T)=\sigma_{\mathcal{B}_T}(T)$, where $\sigma_{\mathcal{B}_T}(T)$ is the spectrum of T in $\mathcal{B}_T$. Let $\lambda\in\sigma(T)$, there exists a character χ_λ on $\mathcal{B}_T$ such that $\chi_\lambda(T)=\lambda$. Since the set of polynomials is dense in $\mathcal{A}(\mathbb{D})$,

$$\chi_\lambda(f(T))=f(\chi_\lambda(T))=f(\lambda),\qquad f\in\mathcal{A}(\mathbb{D})\,.$$

Let now $f\in\mathcal{A}(\mathbb{D})$ such that $f(T)$ be compact and let $\lambda\in\sigma(T)\cap\mathbb{T}$. We have

$$|f(\lambda)|=|\lambda^n f(\lambda)|=|\chi_\lambda(T^n f(T))|\le\|T^n f(T)\|. \tag{2}$$

Since T is in class C_0, $T^n x\to 0$ whenever $x\in H$, (see [11] Proposition III.4.1). Thus for every compact set $C\subset H$,

$$\lim_{n\to\infty}\sup_{x\in C}\|T^n x\|=0.$$

For $C=\overline{f(T)(\mathbb{B})}$, where $\mathbb{B}=\{x\in H\,:\,\|x\|\le 1\}$, we get $T^n f(T)\to 0$. Then it follows from (2) that $f(\lambda)=0$.

(2) $\Rightarrow$ (1) : Without loss of generality, we may assume that $\sigma(T)\cap\mathbb{T}$ is of Lebesgue measure zero. We set

$$\mathcal{I}=\{f\in\mathcal{A}(\mathbb{D})\,:\,f(T)\text{ compact }\};$$

$\mathcal{I}$ is a closed ideal of $\mathcal{A}(\mathbb{D})$. We have to prove that $\mathcal{I}=\mathcal{J}(\sigma(T)\cap\mathbb{T})$. By the Beurling–Rudin theorem, it suffice to show that $Z(\mathcal{I})=\sigma(T)\cap\mathbb{T}$ and $S_{\mathcal{I}}=1$. In the proof of the implication ((1) $\Rightarrow$ (2)) we have seen that $\mathcal{I}\subset\mathcal{J}(\sigma(T)\cap\mathbb{T})$, which implies $\sigma(T)\cap\mathbb{T}\subset Z(\mathcal{I})$. It remains to show that $S_{\mathcal{I}}=1$ and $Z(\mathcal{I})\subset\sigma(T)\cap\mathbb{T}$.

Since T is in class C_0 and I_H-T^*T is compact, by the observation in the beginning of this section $T=U+K$, where U is unitary and K is compact. Moreover we have $\sigma_{wc}(U)=\sigma_{wc}(T)=\sigma(T)\cap\mathbb{T}$ ([14] p. 115), and since $\sigma_{np}(U)$ is countable, we see that $\sigma(U)$ is a subset of $\mathbb{T}$ of Lebesgue measure zero. By the Fatou theorem ([8] p. 80), there exists a nonzero outer function $f\in\mathcal{A}(\mathbb{D})$ which vanishes exactly on $\sigma(U)$. We have $f(U)=0$ since U is unitary. By Lemma 2.2, $f(T)$ is compact. This shows that $S_{\mathcal{I}}=1$ and $Z(\mathcal{I})\subset\sigma(U)$. We shall now show that $Z(\mathcal{I})\subset\sigma_{wc}(U)$.

Let $\lambda\in\sigma_{np}(U)$; λ is an isolated point in $\sigma(U)$ and $\operatorname{Ker}(U-\lambda I_H)$ is of finite dimension. There exists $g\in\mathcal{A}(\mathbb{D})$ with $g(\lambda)\neq 0$ and $f_{|\sigma(U)\setminus\{\lambda\}}=0$. Since $(z-\lambda)f(z)=0$, $z\in\sigma(U)$ and U unitary, $(U-\lambda)f(U)=0$. So $f(U)(H)\subset\operatorname{Ker}(U-\lambda I_H)$. So $f(U)$ is of finite rank, thus $f(U)$ is compact and by Lemma 2.2, $f(T)$ is compact. Hence $\lambda\notin Z(\mathcal{I})$. We deduce that $Z(\mathcal{I})\subset\sigma_{wc}(U)=\sigma_{wc}(T)=\sigma(T)\cap\mathbb{T}$, which finishes the proof.

Let $T \in \mathcal{L}(H)$. The spectral multiplicity of T is the cardinal number given by the formula

$$\mu_T = \inf \ \text{card}\ L,$$

where card L is the cardinal of L and where the infimum is taken over all nonempty sets $L \subset H$ such that span$\{T^n L;\ n \geq 0\}$ is dense in H. Notice that $\mu_T = 1$ means that T is cyclic.

Corollary 2.3. *Let T be a contraction on H with $\mu_T < +\infty$. Assume that there exists a nonzero function $\varphi \in \mathcal{A}(\mathbb{D})$ such that $\varphi(T) = 0$. Then $f(T)$ is compact for every function $f \in \mathcal{A}(\mathbb{D})$ that vanishes on $\sigma(T) \cap \mathbb{T}$.*

Proof. There exists two orthogonal Hilbert subspaces H_u and H_0 that are invariant by T, such that $H = H_u \oplus H_0$, $T_u = T_{|H_u}$ is unitary and $T_0 = T_{|H_0}$ is completely nonunitary (see [11], Theorem 3.2, p. 9 or [13], p. 7). T_0 is clearly in class C_0 and we have $\mu_{T_0} < +\infty$. By Proposition 4.3 of [4], $I_{H_0} - T_0^* T_0$ is compact.

Let $f \in \mathcal{A}(\mathbb{D})$, with $f_{|\sigma(T)\cap\mathbb{T}} = 0$. Since $\sigma(T_0) \subset \sigma(T)$, it follows from Theorem 2.1 that $f(T_0)$ is compact. Now, since T_u is unitary and $\sigma(T_u) \subset \sigma(T) \cap \mathbb{T}$, we get $f(T_u) = 0$. Thus $f(T)$ is compact. □

Remark 2.4. *Let T be a cyclic contraction satisfying condition (1) and with finite spectrum, $\sigma(T) = \{\lambda_1, \ldots, \lambda_n\}$. By Theorem 2 of* [1]*, there exist a function $f = \sum_{n\geq 0} a_n z^n$, $f \neq 0$, such that $\sum_{n\geq 0} |a_n| < +\infty$ and $f(T) = 0$. Then, it follows from Corollary 2.3 that $(T - \lambda_1)\ldots(T - \lambda_n)$ is compact. Thus we find the result Corollary 4.3 of* [2]*, mentioned in the introduction.*

Now we finish this section by showing that the hypothesis "essentially unitary“ in Theorem 2.1 is necessary. Let us first make some observations. An operator $T \in \mathcal{L}(H)$ is called essentially normal if $TT^* - T^*T$ is compact. Notice that if T is a C_0–contraction which is essentially unitary then T^* is essentially unitary too. Hence T is essentially normal since $I_H - T^*T$ and $I_H - TT^*$ are both compacts. Notice also that Theorem 2.1 is of interest in the case of contractions T such that $\sigma(T) \cap \mathbb{T}$ is of Lebesgue measure zero.

Proposition 2.5. *Let $T \in \mathcal{L}(H)$ be a C_0–contraction which is essentially normal and such that $\sigma(T) \cap \mathbb{T}$ is of Lebesgue measure zero. Assume that for every $f \in \mathcal{A}(\mathbb{D})$ vanishing on $\sigma(T) \cap \mathbb{T}$, $f(T)$ is compact. Then T is essentially unitary.*

Proof. Denote by $\mathcal{K}(H)$ the set of all compact operators on H and by $\pi : \mathcal{L}(H) \longrightarrow \mathcal{L}(H)/\mathcal{K}(H)$ the canonical surjection. The essential spectrum $\sigma_{ess}(T)$ of T is defined as the spectrum of $\pi(T)$ in the Banach algebra $\mathcal{L}(H)/\mathcal{K}(H)$.

Let $\lambda \in \sigma(T) \cap \mathbb{D}$. By Fatou theorem [8], there exists a non zero outer function $f \in \mathcal{A}(\mathbb{D})$ such that $f_{|\sigma(T)\cap\mathbb{T}} = 0$. By hypothesis $f(T)$ is compact. The function $z - \lambda$ and f have no common zero in $\overline{\mathbb{D}}$. So there exists two functions g_1 and g_2 in $\mathcal{A}(\mathbb{D})$ such that $(z-\lambda)g_1 + fg_2 = 1$. Thus $(T-\lambda)g_1(T) + f(T)g_2(T) = I_H$, which shows that $\pi(T) - \lambda$ is invertible in $\mathcal{L}(H)/\mathcal{K}(H)$. Hence $\sigma_{ess}(T) \subset \sigma(T) \cap \mathbb{T}$.

By Rudin-Carleson-Bishop theorem (see [8] p. 81), there exists a function $h \in \mathcal{A}(\mathbb{D})$ such that $\overline{z} = h(z),\ z \in \sigma(T) \cap \mathbb{T}$. Since $\pi(T)$ is a normal element in the C^* algebra $\mathcal{L}(H)/\mathcal{K}(H)$, we get $\pi(T)^* = h(\pi(T))$. On the other hand we have $1 - h(z)z = 0$ on $\sigma(T) \cap \mathbb{T}$, which implies that $\pi(I_H) - \pi(T)^*\pi(T) = \pi(I_H) - h(\pi(T))\pi(T) = 0$. Therefore $I_H - T^*T$ is compact. □

3. The case of $f(T)$ for $f \in \mathrm{H}^\infty$

In this section we are interested in the compactness of $f(T)$ when $f \in \mathrm{H}^\infty$. As a consequence of Theorem 2.1 we prove the following result which was first established by Moore–Nordgren in [9], Theorem 1. The proof given in [9] uses a result of Muhly [10] (see Remark 1 below), we give here a simple proof.

Lemma 3.1. *Let T be an essentially unitary C_0–contraction on H, θ be an inner function that divide m_T (i.e $m_T/\theta \in \mathrm{H}^\infty$) and such that $\sigma(\theta) \cap \mathbb{T}$ is of Lebesgue measure zero. Let $\psi \in \mathcal{A}(\mathbb{D})$ be such that $\psi_{|\sigma(\theta)\cap\mathbb{T}} = 0$. If $\phi = \psi m_T/\theta$, then $\phi(T)$ is compact.*

In particular the commutant $\{T\}'$ contains a nonzero compact operator.

Proof. Let $\Theta = m_T/\theta$ and $T_1 = T|_{\overline{\Theta(T)H}}$ be the restriction of T to $\overline{\Theta(T)H}$; T_1 is a C_0–contraction with $m_{T_1} = \theta$. Moreover $I_{H_1} - T_1^*T_1 = P_{H_1}(I_H - T^*T)_{|H_1}$ is compact, where P_{H_1} is the orthogonal projection from H onto H_1. By Theorem 2.1, $\psi(T_1)$ is compact and thus $\phi(T) = \psi(T)\Theta(T) = \psi(T_1)\Theta(T)$ is also compact. □

Lemma 3.2. *Let T be an essentially unitary C_0–contraction on H, θ be an inner function that divide m_T and such that $\sigma(\theta)\cap\mathbb{T}$ is of Lebesgue measure zero. Let $f \in \mathrm{H}^\infty$ be such that $\lim_{n\to+\infty} T^n f(T) = 0$. If $\phi = fm_T/\theta$, then $\phi(T)$ is compact.*

Proof. By the Rudin-Carleson-Bishop theorem, for every nonnegative integers n, there exists $h_n \in \mathcal{A}(\mathbb{D})$ such that $\overline{z}^n = h_n(z), z \in \sigma(\theta) \cap \mathbb{T}$ and $\|h_n\|_\infty = 1$, where $\|.\|_\infty$ is the supremum norm on $\mathbb{T}$ (see [8] p. 81). We have, for every n, $1 - z^n h_n(z) = 0,\ z \in \sigma(\theta) \cap \mathbb{T}$, then by Lemma 3.1, $(I_H - T^n h_n(T))\big(m_T/\theta\big)(T)$ is compact. It follows that $\phi(T) - T^n f(T)h_n(T)\big(m_T/\theta\big)(T)$ is also compact. Since

$$\|T^n f(T)h_n(T)\big(m_T/\theta\big)(T)\| \leq \|T^n f(T)\| \longrightarrow 0,$$

we deduce that $\phi(T)$ is compact. □

We need the following lemma about inner functions, which is in fact contained in the proof of the main result of [15]. For the completeness we include here its proof.

Lemma 3.3. *Let Θ an inner function. There exists a sequence $(\theta_n)_n$ of inner functions such that for each n, θ_n divides Θ, $\sigma(\theta_n) \cap \mathbb{T}$ is of Lebesgue measure zero and for every $z \in \mathbb{D}$, $\lim_{n\to+\infty} \theta_n(z) = \Theta(z)$.*

Proof. Let B_n be the Blaschke product constructed with the zeros of Θ contained in the disk $\{|z| \leq 1 - 1/n\}$, each zero of Θ repeated according to its multiplicity.

Let ν be the singular measure defining the singular part of Θ. There exists $F \subset \mathbb{T}$ of Lebesgue measure Zero such that $\nu(F) = \nu(\mathbb{T})$. There exists a sequence $(K_n)_n$ of compact subsets of F such that $\lim_{n\to\infty} \nu(K_n) = \nu(F)$. For every n, let ν_n be the measure on $\mathbb{T}$ defined by $\nu_n(E) = \nu(E \cap K_n)$. Denote by S_n the singular inner function associated to the measure ν_n. It suffice now to take $\theta_n = B_n S_n$. □

We are now able to prove the main result of this section.

Theorem 3.4. *Let T be an essentially unitary C_0–contraction on H. Let $f \in \mathrm{H}^\infty$. Then the following assertions are equivalents.*

(1) $T^n f(T) \to 0$.

(2) $f(T)$ is compact.

Proof. (1) $\Rightarrow$ (2) : Let $\Theta = m_T$ and let $(\theta_n)_n$ be the sequence of inner functions given by Lemma 3.3. For every n, we set $\varphi_n = m_T/\theta_n$. Since $(\varphi_n)_n$ is a bounded sequence in H^∞ and $\varphi_n(z) \longrightarrow 1$ ($z \in \mathbb{D}$), $(\varphi_n)_n$ converges to 1 uniformly on the compacts of $\mathbb{D}$. Then for every k, there exists a nonnegative integer n_k such that $|\varphi_{n_k}(z)| \geq e^{-1}$ for $|z| \leq \frac{k}{k+1}$. Clearly the sequence $(n_k)_k$ may be chosen to be strictly increasing. Moreover for $|z| \geq \frac{k}{k+1}$, we have $|z^k| \geq e^{-1}$. So

$$e^{-1} \leq |z^k| + |\varphi_{n_k}(z)| \leq 2, \ z \in \mathbb{D}.$$

By he corona theorem ([13], p. 66), there exists two functions h_1 and h_2 in H^∞ such that

$$z^k h_1 + \varphi_{n_k} h_2 = 1 \ \text{ and } \ |h_1|, |h_2| \leq C,$$

where C is an absolute constant. Thus we get

$$T^k f(T) h_1(T) + f(T)\varphi_{n_k}(T) h_2(T) = f(T),$$

and

$$\|T^k f(T) h_1(T)\| \leq C \|T^k f(T)\| \longrightarrow 0.$$

Consequently, $f(T) = \lim_{k\to\infty} f(T)\varphi_{n_k}(T)h_2(T)$ in the $\mathcal{L}(H)$ norm. Finally $f(T)$ is compact since by Lemma 3.2, for every k, $f(T)\varphi_{n_k}(T)h_2(T)$ is compact.

(2) $\Rightarrow$ (1) : see the proof of Theorem 2.1. $\square$

Let T be a contraction on H. It is shown by Esterle, Strouse and Zouakia in [5], that if $f \in \mathcal{A}(\mathbb{D})$, then $\lim_{n\to\infty} T^n f(T) = 0$ if and only if f vanishes on $\sigma(T)\cap\mathbb{T}$. So Theorem 3.4 implies Theorem 2.1. Now, if T is completely non unitary, Bercovici showed in [3] that if $f \in \mathrm{H}^\infty$ and $\lim_{r\to 1-} f(rz) = 0$, for every $z \in \sigma(T)\cap\mathbb{T}$, then $\lim_{n\to\infty} T^n f(T) = 0$. So it follows immediately from this fact and Theorem 3.4 the following result.

Corollary 3.5. *Let T be an essentially unitary C_0–contraction on H. Let $f \in \mathrm{H}^\infty$. If for every $z \in \sigma(T)\cap\mathbb{T}$, $\lim_{r\to 1-} f(rz) = 0$, then $f(T)$ is compact.*

4. Remarks

1. As in Corollary 2.3, Theorem 3.4 holds for a C_0-contraction such that $\mu_T < +\infty$.

2. Let $\mathcal{H}$ be a separable Hilbert space and $\mathrm{L}^2(\mathcal{H})$ the space of weakly measurable functions, $\mathcal{H}$–valued, norm square integrable functions on $\mathbb{T}$. We denote by $\mathrm{H}^2(\mathcal{H})$ the space of functions in $\mathrm{L}^2(\mathcal{H})$ whose negatively indexed Fourier coefficients vanish. The space of bounded weakly measurable function on $\mathbb{T}$ taking values in $\mathcal{L}(\mathcal{H})$ is $\mathrm{L}^\infty(\mathcal{L}(\mathcal{H}))$, and the subspace of $\mathrm{L}^\infty(\mathcal{L}(\mathcal{H}))$ consisting of those functions whose negatively indexed Fourier coefficients vanish is $\mathrm{H}^\infty(\mathcal{L}(\mathcal{H}))$. A function $\Theta \in \mathrm{H}^\infty(\mathcal{L}(\mathcal{H}))$ is called inner if it is unitary valued almost everywhere. The shift S on $\mathrm{H}^2(\mathcal{H})$ is given by $SF(z) = zF(z)$. If $\Theta \in \mathrm{H}^\infty(\mathcal{L}(\mathcal{H}))$ is an inner function , $\Theta\mathrm{H}^2(\mathcal{H})$ is an invariant subspace for S. Let $\mathcal{K}(\Theta)$ be the orthogonal complement of $\Theta\mathrm{H}^2(\mathcal{H})$ and $P_{\mathcal{K}(\Theta)}$ be the orthogonal projection on $\mathcal{K}(\Theta)$. The compression $S(\Theta)$ of the shift S to a subspace $\mathcal{K}(\Theta)$ is given by

$$S(\Theta)F = P_{\mathcal{K}(\Theta)}(SF),$$

If T is a C_0–contraction on a Hilbert H, then there exists a Hilbert space $\mathcal{H}$ and inner functions $\Theta \in \mathrm{H}^\infty(\mathcal{L}(\mathcal{H}))$ such that T is unitarily equivalent to $S(\Theta)$.

Muhly showed in [10] that for $f \in \mathrm{H}^\infty$, the operator $f(T)$ is compact if and only if

$$f\Theta^* \in \mathrm{H}^\infty(\mathcal{L}(\mathcal{H})) + C(S_\infty(\mathcal{H})),$$

where $C(S_\infty(\mathcal{H}))$ is the space of continuous functions on $\mathbb{T}$ that takes values in $S_\infty(\mathcal{H})$, the space of compact operators on $\mathcal{H}$.

Suppose now that the contraction T satisfies the hypothesis of Theorem 3.4. Since $\mathrm{H}^\infty + C(\mathbb{T}) = \mathrm{Clos}_{\,\mathrm{L}^\infty(\mathbb{T})}\{\overline{z}^n\mathrm{H}^\infty : n \geq 0\}$ ([13] p. 183), we get for every $f \in \mathrm{H}^\infty$,

$$\|T^n f(T)\| = \inf_{h\in\mathrm{H}^\infty} \|T^n f(T)+m_T(T)h(T)\| \leq \inf_{h\in\mathrm{H}^\infty} \|z^n f(z)+m_T(z)h(z)\|_\infty$$
$$= \inf_{h\in\mathrm{H}^\infty} \|f(z)\overline{m_T}(z) + \overline{z}^n h(z)\|_\infty = \mathrm{dist}(f\overline{m_T}, \overline{z}^n\mathrm{H}^\infty).$$

So if $f\overline{m_T} \in \mathrm{H}^\infty + C(\mathbb{T})$, then $T^n f(T) \to 0$ and by Theorem 3.4, $f(T)$ is compact. We do not know if the converse is true that is $f(T)$ compact implies that $f\overline{m_T} \in \mathrm{H}^\infty + C(\mathbb{T})$.

3. The following remark gives an example of a C_0–contraction T such that zero is the unique compact operator contained in $\mathrm{H}^\infty(T)$. Let S be any C_0– contraction on H. We consider the Hilbert space

$$\ell^2(H) = \big\{x = (x_n)_{n\geq1} \subset H : \|x\|_2 = \big(\sum_{n\geq1} \|x_n\|^2\big)^{\frac{1}{2}} < +\infty\big\}.$$

and the operator T defined on $\ell^2(H)$ by the formula : $Tx = (Sx_n)_n$, $x = (x_n)_n$. Clearly T is a C_0– contraction with $m_T = m_S$. We claim that for every $f \in \mathrm{H}^\infty$ such that $f(T) \neq 0$, $f(T)$ is not compact. Indeed, let $v \in H$ with norm 1 and such that $f(S)v \neq 0$ and for $n \geq 1$, let $x(n) \in \ell^2(H)$ defined by $x(n)_k = v$ if $k = n$ and $x(n)_k = 0$ if $k \neq n$. Since for every k the projection $x \to x_k$ is continue, zero is the unique limit of any convergent subsequence of $\big(f(T)x(n)\big)_n$. On the other hand, for every n, $\|f(T)x(n)\|_2 = \|f(S)v\|$. So all subsequences of $\big(f(T)x(n)\big)_n$ diverge.

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CMI, LATP, Université de Provence, 39, rue F. Joliot-Curie, 13453 Marseille cedex 13, France

E-mail address: kellay@cmi.univ-mrs.fr

Université de Bordeaux, UMR 5251, 351, cours de la Libération, F-33405 Talence cedex, France

E-mail address: Mohamed.Zarrabi@math.u-bordeaux1.fr